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ANALYSIS OF ADHESIVELY BONDED JOINTS WITH GAPS SUBJECTED TO BENDING

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Abstract—The two dimensional plane strain problem of adhesively bonded joints with gap defects in the adhesive is treated. A simple lap joint with a gap is considered, and the effect of bending is included in the formulation. The results show steep edge gradients for peel and shear stresses, and the peak stresses always occur at the extreme ends of the overlap, regardless of the presence of the gaps. The gap has little effect on the peak stresses, unless it is sufficiently close to an end, where an example indicates that stresses can be affected by as much as 25%. Published by Elsevier Science Ltd.

INTRODUCTION

Adhesively bonded structures are used in aerospace and high technology structural components and have many advantages in terms of design flexibility, cost reduction and simplicity of fabrication. It is known however, that defects in the adhesive can severely reduce the bond strength. The presence of gaps or disbond type of flaws in the adhesive will increase the peak stress levels which occurs at the joint ends and near the defect itself. The joint may fail at the ends of the joint at the ultimate stress or it may fail under cyclic loading where local debonding near the flaw can grow.

In past work related to the present study Delale *et al.* (1981) considered problems for a continuous joint and included bending and transverse shear in the adherends. With appropriate assumptions, they reduced the problem to a system of differential equations, which is solved in closed form. The validity of their method was established by comparing the closed form results of a sample problem with those of a finite element analysis. Renton and Vinson (1978) obtained similar behavior for the shear and peel stresses. Their closed form solution, in conjunction with their BOND4 computer program, represents a rather complete analysis of the single lap joint, where both transverse shear and normal strains were also included in the adherends.

The initial work on the effect of bondline flaws and gaps in the adhesive, as was done by Hart-Smith (1981), Kan and Ratwani (1983), and Rossettos *et al.* (1993, 1994) used the essence of the shear lag model, where the adherends take on only axial load and the adhesive takes only shear (see Erdogan and Ratwani (1971)). This is appropriate in bonded joints which are designed so that the net load path does not produce bending.

In the present paper, the two dimensional plane strain problem of adhesively bonded joints is considered. The structure consists of two different adherends bonded by an adhesive layer which contains a gap (void), and the joint undergoes bending. The analysis is a variation of the method given by Delale *et al.* (1981), where simplifying assumptions consider the adherends to be orthotropic plates for which a transverse shear theory is used. It is also assumed that the thicknesses of the adherends are constant, and are small compared to the lateral dimensions of the bonded region. It is further assumed that the thickness of the adhesive is sufficiently small so that the thickness variation of normal and shear stresses in the adhesive can be neglected. The problem in the present work is then reduced to a sixth order differential equation in terms of the stress resultant in one of the adherends and is solved in closed form. Once the load distribution in one of the adherends is known, the shear and normal stresses (peel stresses) can be determined. Note that the formulation used here is different than the one used by Delale *et al.* (1981), where the governing differential equations are given in terms of shear and peel stresses in the adhesive.

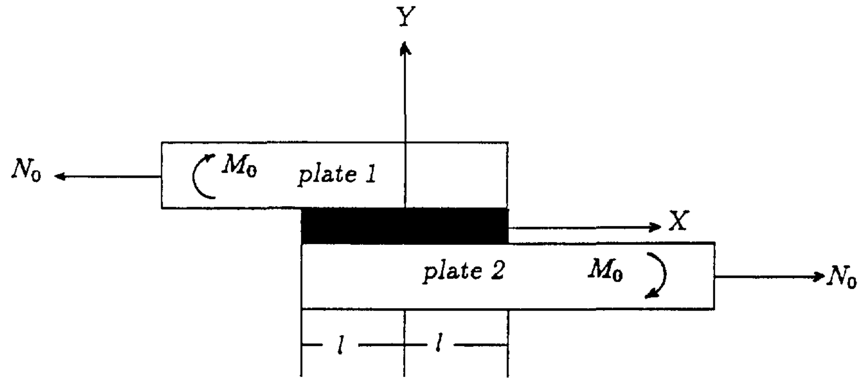

Fig. 1. Continuous single lap joint under tension.

The problem of adhesively bonded joints with a gap in the adhesive, and which undergoes bending, is treated for the first time. The analysis is similar to the continuous bond problem, with the exception that the governing equations in the gap region will be different, and additional continuity conditions connecting the gap region with the continuous adhesive are used to obtain the solution. The continuity conditions in the present approach are conveniently described in terms of the stress and moment resultants.

ANALYSIS

Analysis for a continuous joint

Consider an adhesively bonded structure which consists of two plates bonded by an adhesive layer. Plate 1 is a composite with orthotropic characteristics and plate 2 is made of an isotropic material (Fig. 1).

Let $N_1(x)$ and $N_2(x)$ be the resultant forces per unit width in plate 1 and plate 2 respectively. $Q_1(x)$, $Q_2(x)$ and $M_1(x)$, $M_2(x)$ are the transverse shear load and moment per unit width in plates 1 and 2 respectively. From Fig. 3, the following can be obtained

$$\frac{dN_1}{dx} = \tau \tag{1}$$

$$\frac{dN_2}{dx} = -\tau \tag{2}$$

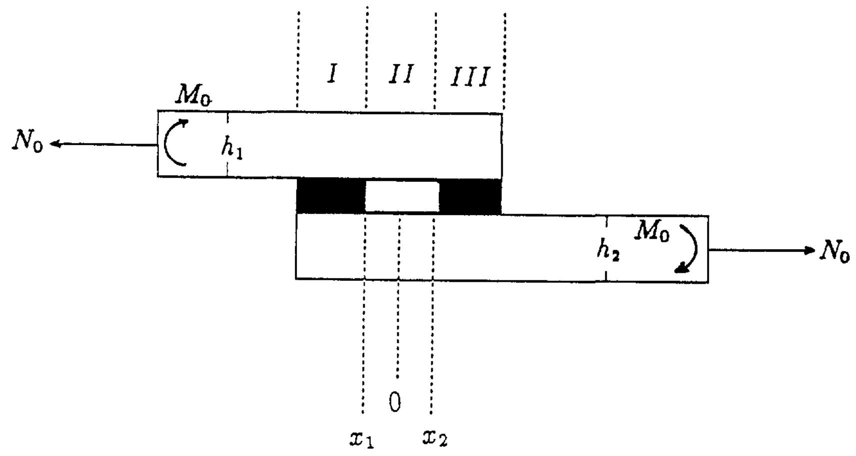

Fig. 2. Single lap joint with a gap (void) under tension.

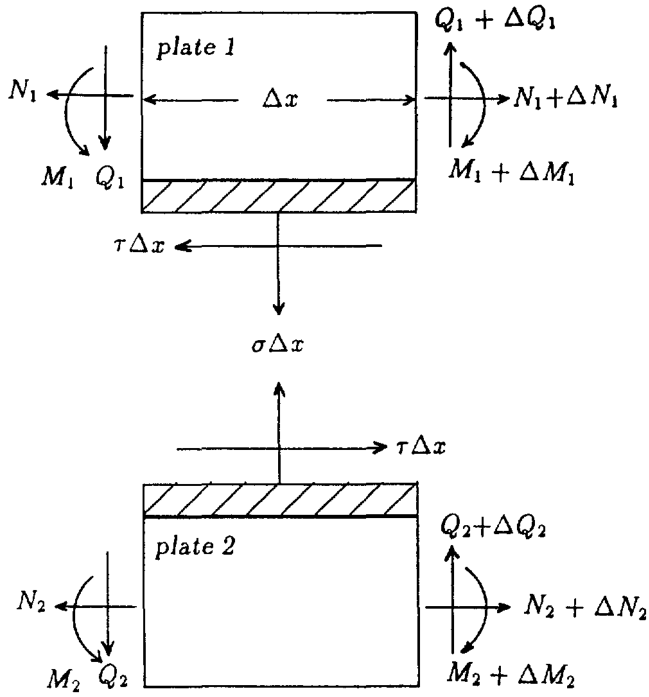

Fig. 3. Free body diagram of plate 1 and plate 2 (continuous).

$$\frac{dQ_1}{dx} = \sigma \tag{3}$$

$$\frac{dQ_2}{dx} = -\sigma \tag{4}$$

$$\frac{dM_1}{dx} = Q_1 - \frac{h_1+h_0}{2}\tau \tag{5}$$

$$\frac{dM_2}{dx} = Q_2 - \frac{h_2+h_0}{2}\tau \tag{6}$$

where h_1 and h_2 are the thickness of adherend 1 and adherend 2 respectively and h_0 is the thickness of the adhesive. Also $\tau = \tau_{xy}$ is the shear stress in the adhesive and $\sigma = \sigma_y$ is the normal (peel) stress in the adhesive. Note that N_1, N_2, M_1, M_2 and Q_1, Q_2 are the load, moment resultant and the shear force per unit width in plate 1 and 2 respectively.

The stress (N_1, N_2, Q_1, Q_2) and moment resultants (M_1, M_2) can be related to the x and y components of the displacements v_i, u_i and to the rotation β_i by the following:

$$\frac{du_1}{dx} = C_1 N_1 \tag{7}$$

$$\frac{du_2}{dx} = C_2 N_2 \tag{8}$$

$$\frac{dv_1}{dx} + \beta_1 = \frac{Q_1}{B_1} \tag{9}$$

$$\frac{dv_2}{dx} + \beta_2 = \frac{Q_2}{B_2} \tag{10}$$

$$\frac{d\beta_1}{dx} = D_1 M_1 \tag{11}$$

$$\frac{d\beta_2}{dx} = D_2 M_2 \tag{12}$$

where $B_i = \frac{5}{6} h_i G_i$ and $C_i = (1 - \nu_{ix}\nu_{iz})/h_i E_i$ and $D_i = 12(1 - \nu_{ix}\nu_{iz})/h_i^3 E_i$. Note that E_i, G_i ($i = 1$, 2) are the modulus of elasticity and rigidity of the adherends respectively.

To obtain the strains ε_x, ε_y and γ_{xy} in the adhesive, the y-dependence of the strains is assumed to be negligible. The strain ε_x is an average in plane strain. The following relations are used.

$$\varepsilon_y = \frac{(v_1 - v_2)}{h_0} \tag{13}$$

$$\varepsilon_x = \frac{1}{2}\left(\frac{du_1}{dx} + \frac{du_2}{dx} - \frac{h_1}{2}\frac{d\beta_1}{dx} + \frac{h_2}{2}\frac{d\beta_2}{dx}\right) \tag{14}$$

$$\gamma_{xy} = \frac{1}{h_0}\left(u_1 - u_2 - \frac{h_1}{2}\beta_1 - \frac{h_2}{2}\beta_2\right). \tag{15}$$

The stress-strain relations are then given by

$$\varepsilon_y = T\sigma - Z\varepsilon_x \tag{16}$$

$$\gamma_{xy} = \frac{\tau}{G} \tag{17}$$

where $T = (1 - \nu - 2\nu^2)/E(1 - \nu)$ and $Z = \nu/(1 - \nu)$. Also note that E, G and ν are the elastic constants of the adhesive.

By eliminating the displacement and strain quantities in eqns (1–17) in terms of stress resultants and moments by appropriate differentiation and combination, two coupled differential equations in terms of N_1 and M_1 can be obtained (derivation details are given by Olia (1992)). They are written as follows:

$$\frac{d^4 M_1}{dx^4} + f_1 \frac{d^4 N_1}{dx^4} + f_2 \frac{d^2 M_1}{dx^2} + f_3 \frac{d^2 N_1}{dx^2} + f_4 M_1 + f_5 N_1 = f_6 M_0 \tag{18}$$

$$\frac{d^2 N_1}{dx^2} + k_1 N_1 + k_2 M_1 + k_3 N_0 + k_4 M_0 = 0 \tag{19}$$

where N_0 is the load applied to the plates and $M_0 = \left(\frac{h_0}{2} + \frac{h_1 + h_2}{4}\right) N_0$ is the moment needed to maintain equilibrium, and where

$$f_1 = H_1;\quad f_2 = \left(\frac{h_1 D_1 Z}{4T} - \frac{1}{Th_0 B_1} + \frac{h_2 D_2 Z}{4T} - \frac{1}{Th_0 B_2}\right)$$

$$f_3 = \left(\frac{-ZC_1}{2T} + \frac{ZC_2}{2T} - \frac{1}{Th_0}\left(\frac{H_1}{B_1} - \frac{H_2}{B_2}\right) + \frac{h_2 D_2 Z H_3}{4T} - \frac{H_3}{Th_0 B_2}\right)$$

$$f_4 = \frac{D_1 + D_2}{Th_0};\quad f_5 = \frac{H_3 D_2}{Th_0};\quad f_6 = \frac{D_2}{Th_0}$$

$$k_1 = -\left(\frac{GC_1 + GC_2}{h_0} + \frac{Gh_2 D_2 H_3}{2h_0}\right);\quad k_2 = \frac{G(h_1 D_1 - h_2 D_2)}{2h_0}$$

$$k_3 = \frac{GC_2}{h_0};\quad k_4 = \frac{Gh_2 D_2}{2h_0}$$

and

$$H_1 = \frac{h_1 + h_0}{2};\quad H_2 = \frac{h_2 + h_0}{2};\quad H_3 = h_0 + \frac{h_1 + h_2}{2}.$$

Equation (19) is used to eliminate M_1 from eqn (18) which then becomes the following sixth order differential equation in terms of N_1,

$$\frac{d^6 N_1}{dx^6} + \alpha_1 \frac{d^4 N_1}{dx^4} + \alpha_2 \frac{d^2 N_1}{dx^2} + \alpha_3 N_1 = \beta_1 N_0 + \beta_2 M_0 \tag{20}$$

where

$$\alpha_1 = k_1 - f_1 k_2 + f_2;\quad \alpha_2 = f_2 k_1 - f_3 k_2 + f_4;\quad \alpha_3 = f_4 k_1 - f_5 k_2$$
$$\beta_1 = -f_4 k_3;\quad \beta_2 = -f_6 k_2 - f_4 k_4.$$

Equation (20) is the governing differential equation for plate 1 in terms of N_1. A solution can be assumed in the form of $N_1 = e^{Rx}$ for the homogeneous part. The complete solution can then be written as

$$N_1 = \sum_{i=1}^{6} C_i e^{R_i x} + \frac{\beta_1}{\alpha_3} N_0 + \frac{\beta_2}{\alpha_3} M_0 \tag{21}$$

where the R_i are roots of the characteristic equation associated with the homogeneous part of eqn (20). Note that the terms involving N_0 and M_0 on the right hand side of eqn (21) are the particular solution. Once the load in plate 1 (N_1) is known, general solutions for all other stress and moment resultants in plates 1 and 2 can be obtained as follows:

$$N_2 = -\sum_{i=1}^{6} C_i e^{R_i x} + \left(1 - \frac{\beta_1}{\alpha_3}\right) N_0 - \frac{\beta_2}{\alpha_3} M_0 \tag{22}$$

$$M_1 = -\frac{1}{k_2}\left(\sum_{i=1}^{6} (R_i^2 + k_1) C_i e^{R_i x} + \left(\frac{k_1 \beta_1}{\alpha_3} + k_3\right) N_0 + \left(\frac{k_1 \beta_2}{\alpha_3} + k_4\right) M_0\right) \tag{23}$$

and since $M_2 = -H_3 N_1 - M_1 + M_0$ then

$$M_2 = \sum_{i=1}^{6}\left(\frac{R_i^2+k_1}{k_2}-H_3\right)C_i e^{R_i x} + \left[\frac{\dfrac{k_1\beta_1}{\alpha_3}+k_3}{k_2}-\frac{H_3\beta_1}{\alpha_3}\right]N_0 + \left[\frac{\dfrac{k_1\beta_2}{\alpha_3}+k_4}{k_2}-\frac{H_3\beta_2}{\alpha_3}\right]M_0 \tag{24}$$

$$Q_1 = \sum_{i=1}^{6}\left(H_1 R_i - \frac{R_i^3+k_1 R_i}{k_2}\right)C_i e^{R_i x} \tag{25}$$

$$Q_2 = \sum_{i=1}^{6}\left((H_2-H_3)R_i + \frac{R_i^3+k_1 R_i}{k_2}\right)C_i e^{R_i x}. \tag{26}$$

Boundary conditions

To find the constants (C_i), the following boundary conditions can be used. At $x = l$ [see Fig. (1)] ($2l$ is the length of the adhesive)

$$N_1 = 0 \quad Q_1 = 0 \quad M_1 = 0 \tag{27}$$

and at $x = -l$

$$N_2 = 0 \quad Q_2 = 0 \quad M_2 = 0. \tag{28}$$

Once the C_i's are determined, the shear and normal (peel) stresses in the adhesive can be found using eqns (21–26) and (1–4)

Analysis of a joint with gaps (voids)

In order to obtain the governing equations for a simple lap joint with a gap or void, the adhesive is divided into three regions, as shown in Fig. 2. The equilibrium equations (1–6) are valid for region one and region three. In the void region, the shear stress $\tau_{xy} = 0$ and normal stress $\sigma_y = 0$, as a result the equations for region two (gap or void) are (Fig. 4)

$$\frac{dN_1}{dx} = \tau = 0 \quad \frac{dN_2}{dx} = -\tau = 0 \tag{29}$$

$$\frac{dQ_1}{dx} = \sigma = 0 \quad \frac{dQ_2}{dx} = -\sigma = 0 \tag{30}$$

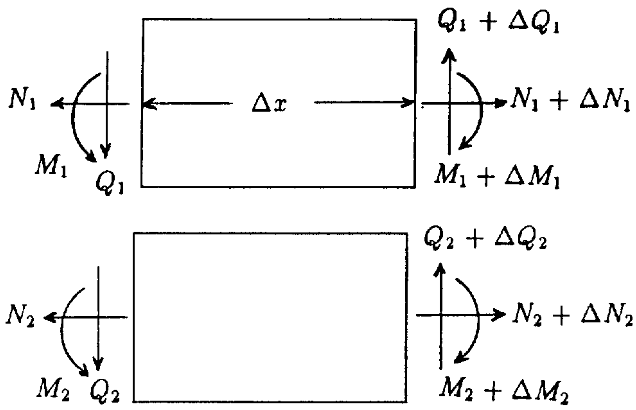

Fig. 4. Free body diagram of plate 1 and plate 2 (gap).

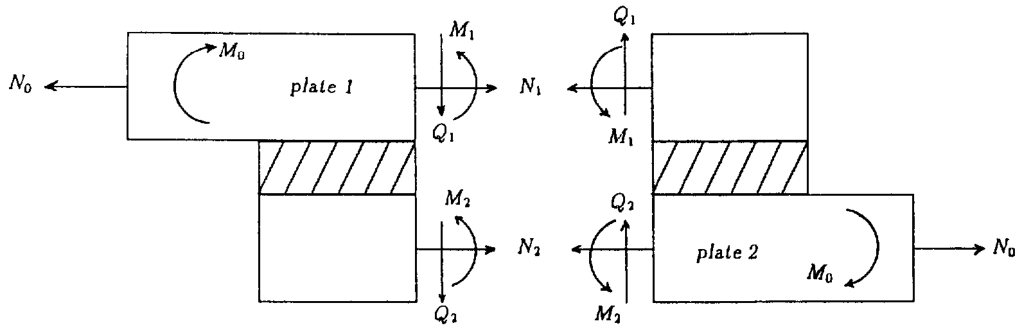

Fig. 5. Overall equilibrium diagram of a single lap joint.

$$\frac{dM_1}{dx} = Q_1 \quad \frac{dM_2}{dx} = Q_2. \tag{31}$$

Note that for any value of x the overall equilibrium gives (Fig. 5)

$$N_1 + N_2 = N_0 \quad Q_1 + Q_2 = 0 \quad M_1 + M_2 = -H_3 N_1 + M_0. \tag{32}$$

The solution to N_1, N_2, M_1, M_2, Q_1, Q_2 in region one and three are obtained using eqns (21–26), except that the constants C_i's are different in each region. Using eqns (29–31) and overall equilibrium, N_1, N_2, M_1, M_2, Q_1, Q_2 in region two (gap) can be easily obtained. Therefore in region two, the following holds

$$N_{1II} = A \quad N_{2II} = N_0 - A \tag{33}$$

$$Q_{1II} = B \quad Q_{2II} = B \tag{34}$$

$$M_{1II} = Bx + D \quad M_{2II} = -H_3 A - Bx - D + M_0 \tag{35}$$

where A, B and D are constants and can be determined together with C_i's by using the appropriate boundary and continuity conditions.

Boundary conditions

As was mentioned earlier the adhesive is divided into three regions (*I*, *II*, *III*).

In region *I*, at $x = -l$

$$N_{2I}(-l) = 0 \quad Q_{2I}(-l) = 0 \quad M_{2I}(-l) = 0 \tag{36}$$

and in region *III*, at $x = l$

$$N_{1III}(l) = 0 \quad Q_{1III}(l) = 0 \quad M_{1III}(l) = 0. \tag{37}$$

Note that in the above equations, the first subscript represents the adherend, while the second subscript represents the region number.

There are 6 continuity conditions which are given as follows:

At $x = x_1$ (interface of region *I* and *II*)

$$N_{1I}(x_1) = N_{1II}(x_1) \quad Q_{1I}(x_1) = Q_{1II}(x_1) \quad M_{1I}(x_1) = M_{1II}(x_1) \tag{38}$$

and at $x = x_2$ (interface of region *II* and *III*)

$$N_{1II}(x_2) = N_{1III}(x_2) \quad Q_{1II}(x_2) = Q_{1III}(x_2) \quad M_{1II}(x_2) = M_{1III}(x_2). \tag{39}$$

There are 15 constants of integration (6 in region *I*, 6 in region *III*, 3 in region *II*), therefore to obtain the solution for all regions, 3 more conditions are needed, which are as follows. (See Appendix for the details of the derivation.)

$$\left(\frac{dN_{1III}}{dx}\right)_{x_2^+} - \left(\frac{dN_{1I}}{dx}\right)_{x_1^-} = \frac{G}{h_0}\left(\int_{x_1}^{x_2}\left(C_1N_{1II} - \frac{h_1}{2}D_1M_{1II}\right)dx - \int_{x_1}^{x_2}\left(C_2N_{2II} - \frac{h_2}{2}D_2M_{2II}\right)dx\right) \tag{40}$$

$$\left(\frac{dM_{1III}}{dx}\right)_{x_2^+} - \left(\frac{dM_{1I}}{dx}\right)_{x_1^-} = Q_{1II}(x_2) - Q_{1II}(x_1) - \frac{G(h_1+h_0)}{2h_0}\left(\int_{x_1}^{x_2}\left(C_1N_{1II} - \frac{h_1}{2}D_1M_{1II}\right)dx - \int_{x_1}^{x_2}\left(C_2N_{2II} - \frac{h_2}{2}D_2M_{2II}\right)dx\right) \tag{41}$$

and

$$\left(\frac{d^3Q_{1III}}{dx^3}\right)_{x_2^+} - \left(\frac{d^3Q_{1I}}{dx^3}\right)_{x_1^-} = \left(\frac{1}{Th_0}(D_2M_2 - D_1M_1)_{x_2} - \frac{1}{Th_0}(D_2M_2 - D_1M_1)_{x_1}\right). \tag{42}$$

Results and discussion

The method developed in this paper is used to analyze a simple lap joint with a gap in the adhesive. Using this method for the case with no gap (void) produces results identical to the ones given by Delale *et al.* (1981), validating the present approach. A comparison of the results obtained by each approach is given in Table 1. The properties used in present calculations for the adherends and adhesive are given here as follows. Adherend 1 is boron-epoxy orthotropic plate ($E_{1x} = 3.24 \times 10^7$ psi, $E_{1z} = 3.5 \times 10^6$ psi, $G_1 = 1.23 \times 10^6$ psi, $\nu_{1x} = 0.23$, $h_1 = 0.03$ in.), adherend 2 is aluminium plate ($E_2 = 10^7$ psi, $\nu_2 = 0.3$, $h_2 = 0.09$ in.) and the adhesive is made of epoxy ($E = 4.45 \times 10^5$ psi, $G = 1.65 \times 10^5$ psi, $h_0 = 0.004$ in.).

Table 1. Comparison of present results with Delale *et al.* (1981)

	Present results		Delale *et al.* (1981)	
X	$\bar{\tau}$	$\bar{\sigma}$	$\bar{\tau}$	$\bar{\sigma}$
−1.00	−13.4362841	41.1925850	−13.436	41.191
−0.90	−3.6897621	−4.2951283	−3.690	−4.295
−0.80	−0.8575040	−1.7132319	−0.858	−1.713
−0.70	−0.1411121	−0.5552792	−0.141	−0.555
−0.60	0.0112195	−0.1832258	0.011	−0.183
−0.50	0.0281249	−0.0629277	0.028	−0.063
−0.40	0.0198290	−0.0227094	0.020	−0.023
−0.30	0.0110012	−0.0087121	0.011	−0.009
−0.20	0.0051124	−0.0036940	0.005	−0.004
−0.10	0.0012140	−0.0020126	0.001	−0.002
0.00	−0.0022779	−0.0018766	−0.002	−0.002
0.10	−0.0072486	−0.0029729	−0.007	−0.003
0.20	−0.0166439	−0.0058942	−0.017	−0.006
0.30	−0.0361454	−0.0123793	−0.036	−0.012
0.40	0.0775971	−0.0263139	−0.078	−0.026
0.50	−0.1662305	−0.0560145	−0.166	−0.056
0.60	−0.3561518	−0.1190566	−0.356	−0.119
0.70	−0.7638092	−0.2521242	−0.764	−0.252
0.80	−1.6409304	−0.5259318	−1.641	−0.526
0.90	−3.5379338	−0.8505507	−3.538	−0.851
1.00	−7.8056602	9.6929560	−7.806	9.693

In the results that follow, $\bar{\sigma}$ and $\bar{\tau}$ are nondimensional peel stress and shear stress in the adhesive respectively, and they are defined as follows:

$$\bar{\sigma} = \sigma\left(\frac{2l}{N_0}\right) \quad \bar{\tau} = -\tau\left(\frac{2l}{N_0}\right). \tag{43}$$

Figure 6 shows the peel stress $\bar{\sigma}$ vs the overlap length (X). Adherends 1 and 2 are identical (boron-epoxy laminates) and the adhesive (epoxy) length is 1.0 in. The gap (void) is symmetric and its length is 0.2 in. It can be seen that the gap will increase the value of peel stress at the interface of the gap and the bond, but the peel stress $\bar{\sigma}$ will increase slightly at the edges compared to the case without the gap. The shear stress is plotted against the position along the overlap in Fig. 7 for the case with no gap (void) and the case with a gap (void) present. The length of the gap is 0.3 in. and is close to the edge. Figure 7 shows that the shear stress is essentially unaffected far from the void (left edge), when compared to the case with no void. The inclusion of bending in the formulation seems to create a narrower boundary layer near the edges of the overlap compared to the shear lag model results obtained by Rossettos *et al.* (1993, 1994), where the joint is rigid in bending. It should be pointed out that since the joint used in this analysis is not rigid in bending, and the problem is formulated with bending (rotation) considered, more deformation mechanisms are involved and energy is divided between peel and shear strain energies. Figure 8 shows the peel stress $\bar{\sigma}$ as a function of distance along the overlap length (X). The gap (void) is 0.1 in. It can be seen that the effect of the gap on the peel stress is a largely steep edge effect. The gap appears to relieve the stresses at the edge of the overlap (near the gap), while the peel stress increases at the edge of the gap itself compared to the no gap case. The same trend can be seen in Fig. 9, where the gap (void) length is larger (0.3 in.). Figure 10 shows that the peak peel stresses decrease 40% by increasing the adhesive thickness. Note that for Figs 7–10, the curves are non-symmetric due to bending. In all cases, regardless of the

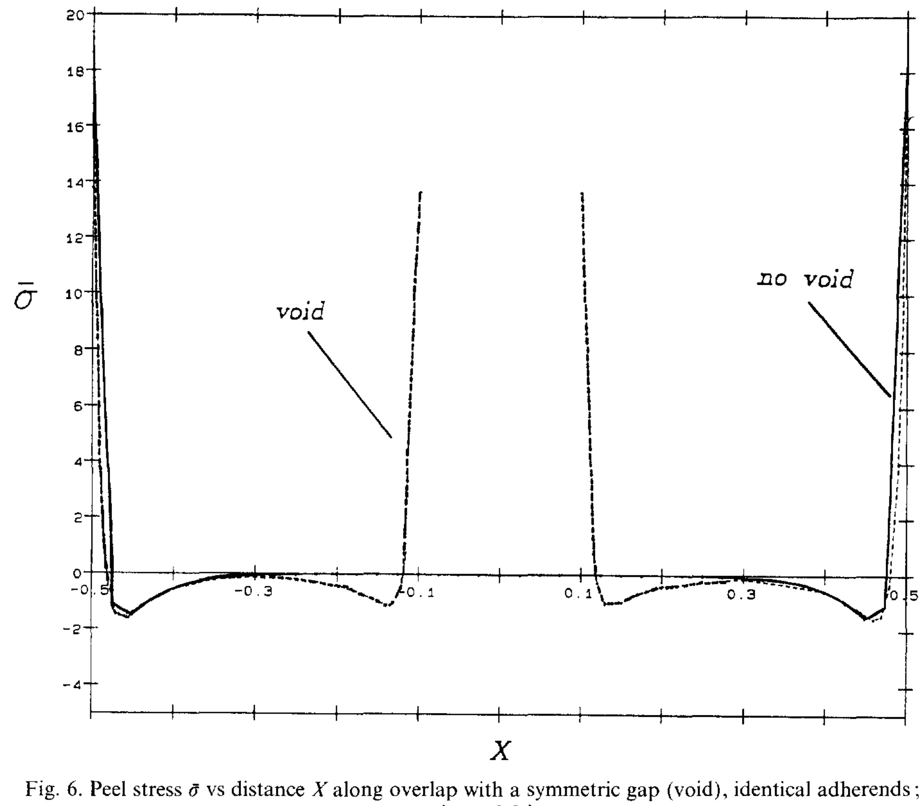

Fig. 6. Peel stress $\bar{\sigma}$ vs distance X along overlap with a symmetric gap (void), identical adherends; gap size = 0.2 in.

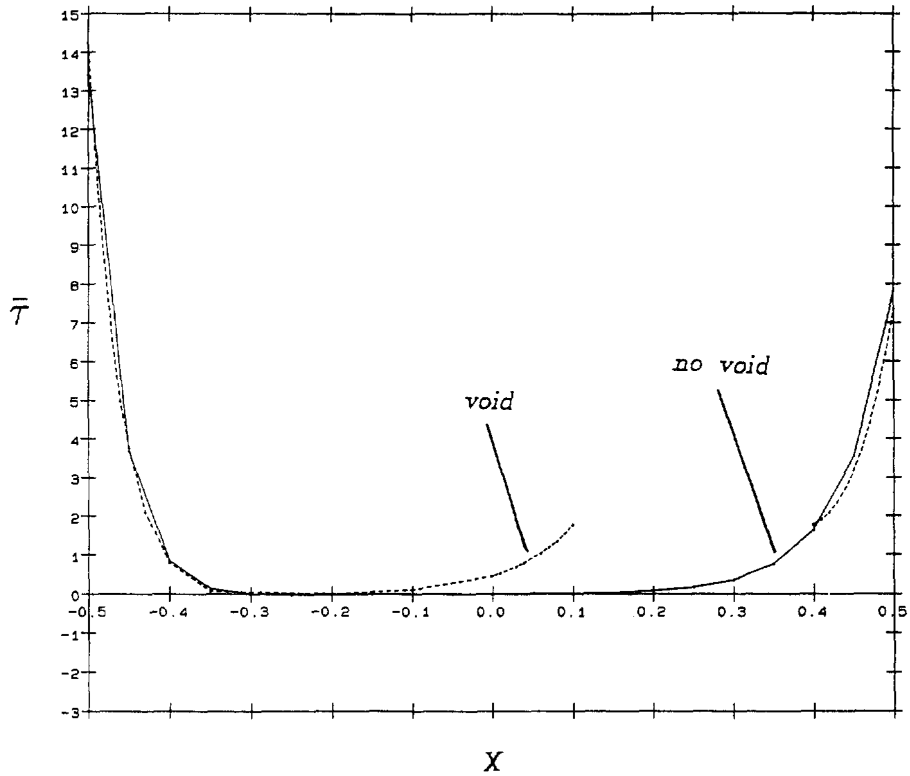

Fig. 7. Shear stress $\bar{\tau}$ vs distance X along overlap with a gap and no gap; gap (void) size = 0.3 in.

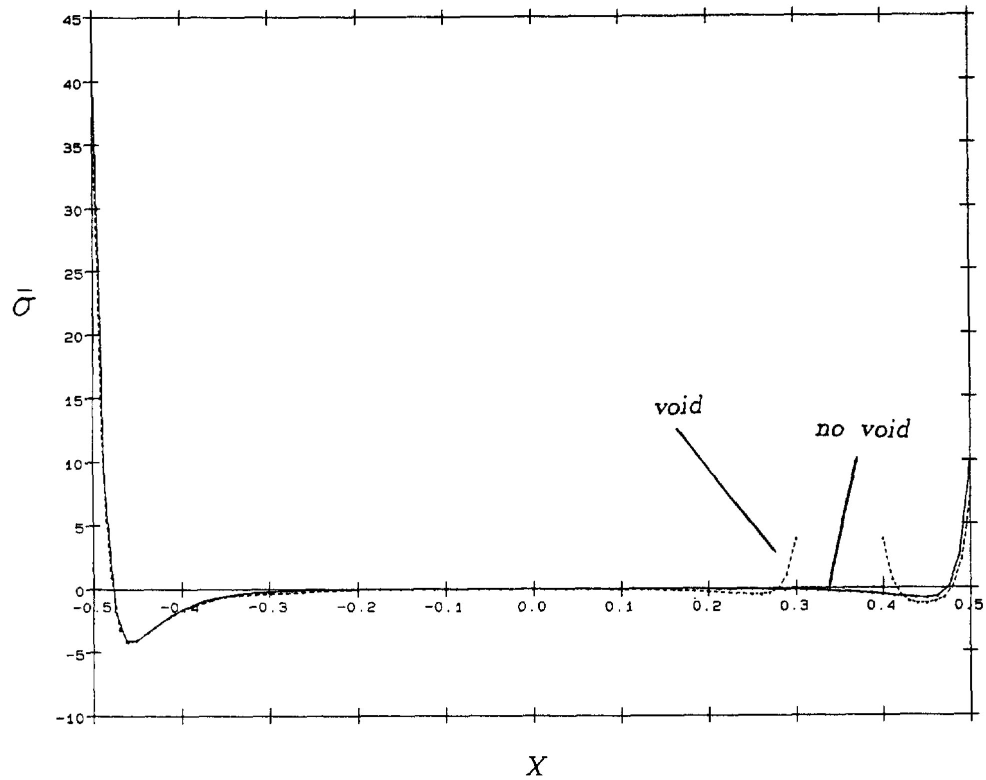

Fig. 8. Peel stress $\bar{\sigma}$ vs distance X along overlap with a gap and no gap; gap (void) size = 0.1 in.

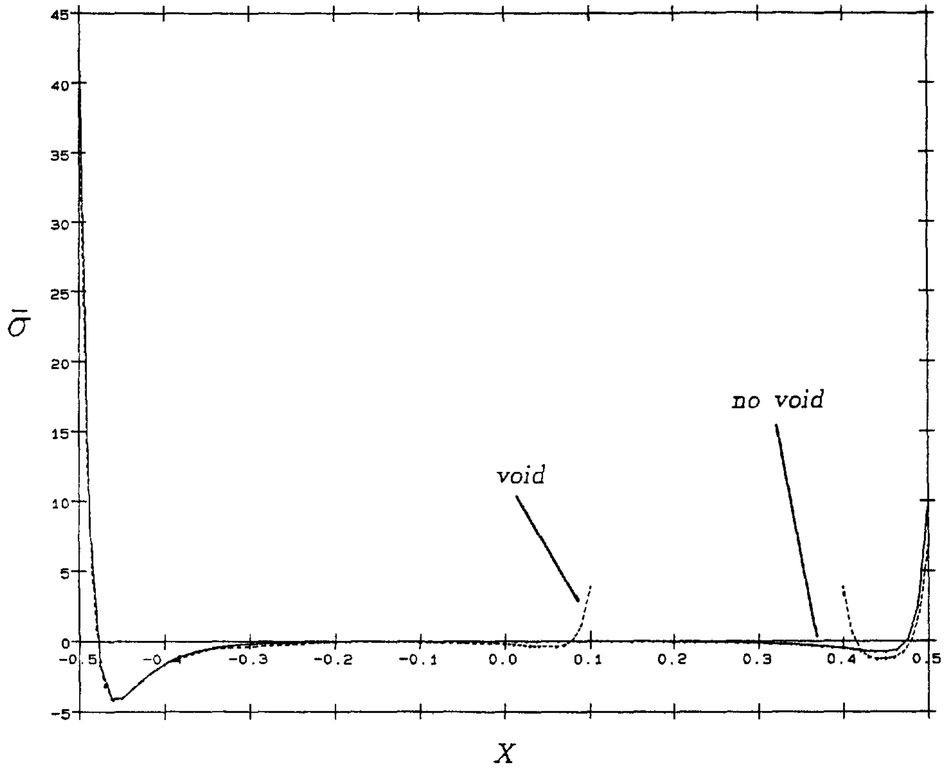

Fig. 9. Peel stress $\bar{\sigma}$ vs distance X along overlap with a gap and no gap; gap (void) size = 0.3 in.

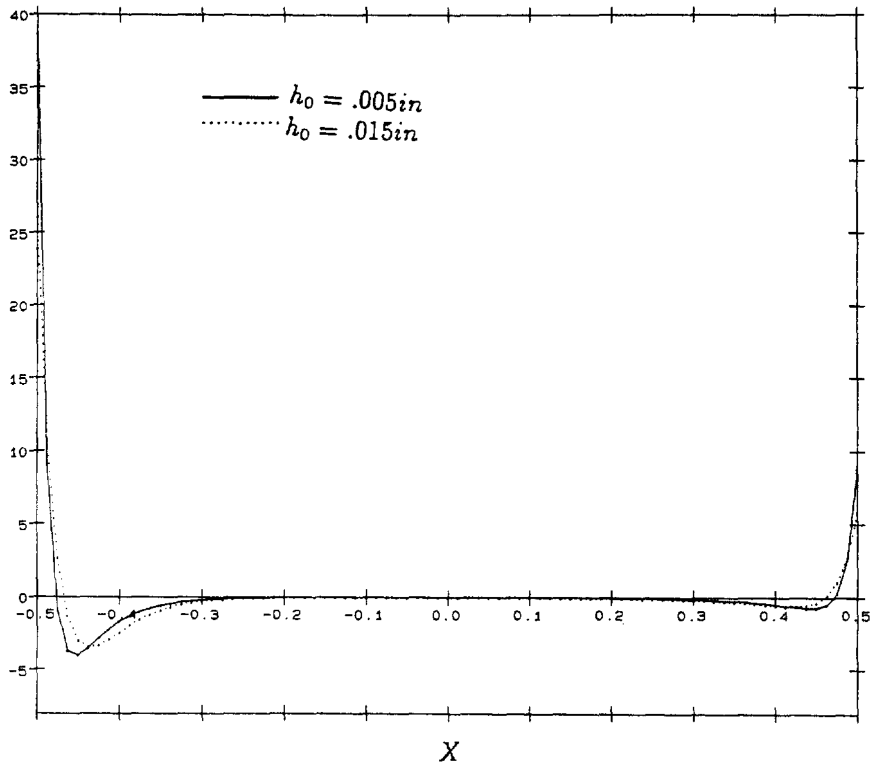

Fig. 10. Peel stress $\bar{\sigma}$ vs distance X along overlap for different adhesive thickness with no gap.

presence of gaps (voids), the peak stresses always occur at the extreme ends of the adhesive bond.

CONCLUSIONS

A convenient method is developed to analyze a simple lap joint with a gap (void) defect in the adhesive, and undegoing general bending deformation. It is limited to a two dimensional plane strain analysis. The method is a variation of an approach used by Delale *et al.* (1981) for the continuous joint. Using the present method for the case with no gap produces results identical to those of Delale *et al.* (1981). The present method is particularly useful in handling the boundary and continuity conditions for problems with gap (void) defects in the adhesive. The solution procedure is applicable to adhesively bonded plates which are assumed to be orthotropic and isotropic. The results show that inclusion of bending in the formulation creates a narrower boundary layer compared to the shear lag model (see e.g. Rossettos *et al.* (1993, 1994)) and stresses are essentially unaffected far from the void. If the void is sufficiently close to the end of the overlap, the stresses can be affected as much as 25%. The results also show that the effect of the gap (void) on the peel and shear stresses are largely steep edge effects. The gap appears to relieve the stresses at the edge of the overlap (near the gap), while the peel and shear stresses increase at the edge of the void itself. One should note that since the loading is not centric and because bending is considered, more deformation mechanisms are involved, which affect the stresses. Regardless of the presence of gaps (voids), the results show that the most critical stress state occurs at the extreme ends of the adhesive bond.

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APPENDIX A

Continuity conditions over gap (void) region

To derive the continuity equations needed over the gap (void) region, we combine the equilibrium equations and stress-strain equations.

To derive eqn (40), the following can be done. Consider the strains

$$\varepsilon_{1x} = \left(\frac{du_1}{dx} - \frac{h_1}{2}\frac{d\beta_1}{dx}\right) \tag{A1}$$

$$\varepsilon_{2x} = \left(\frac{du_2}{dx} - \frac{h_2}{2}\frac{d\beta_2}{dx}\right) \tag{A2}$$

and the axial strain in the adhesive ε_x is the average of the axial strains in plate 1 and 2, so

$$\varepsilon_x = \frac{\varepsilon_{1x} + \varepsilon_{2x}}{2}. \tag{A3}$$

Using eqns (A1) and (A2), and integrating

$$\int_{x_1}^{x_2} \varepsilon_{1x}\, dx = \int_{x_1}^{x_2} du_1 - \frac{h_1}{2}\int_{x_1}^{x_2} d\beta_1 \tag{A4}$$

$$\int_{x_1}^{x_2} \varepsilon_{2x}\, dx = \int_{x_1}^{x_2} du_2 - \frac{h_2}{2}\int_{x_1}^{x_2} d\beta_2 \tag{A5}$$

results in

$$\int_{x_1}^{x_2} \varepsilon_{1x}\, dx = u_1(x_2) - u_1(x_1) - \frac{h_1}{2}[\beta_1(x_2) - \beta_1(x_1)] \tag{A6}$$

$$\int_{x_1}^{x_2} \varepsilon_{2x}\, dx = u_2(x_2) - u_2(x_1) - \frac{h_2}{2}[\beta_2(x_2) - \beta_2(x_1)]. \tag{A7}$$

Equations (A6) and (A7) involve the change in the displacements at the adherend/adhesive interface as we go from x_1^- to x_1^+ over the gap region.

Using eqns (7–8) and (11–12), the following can be obtained.

$$\int_{x_1}^{x_2} \left(C_1 N_1 - \frac{h_1}{2} D_1 M_1\right) dx = u_1(x_2) - u_1(x_1) - \frac{h_1}{2}[\beta_1(x_2) - \beta_1(x_1)] \tag{A8}$$

$$\int_{x_1}^{x_2} \left(C_2 N_2 - \frac{h_2}{2} D_2 M_2\right) dx = u_2(x_2) - u_2(x_1) - \frac{h_2}{2}[\beta_2(x_2) - \beta_2(x_1)] \tag{A9}$$

Using eqns (15), (17) and (1–2), the load in the two plates can be related to axial displacements and rotation in the following

$$\left(\frac{dN_1}{dx}\right)_{x_2^+} = \frac{G}{h_0}\left(u_1 - u_2 - \frac{h_1}{2}\beta_1 - \frac{h_2}{2}\beta_2\right)_{x_2} \tag{A10}$$

$$\left(\frac{dN_1}{dx}\right)_{x_1^-} = \frac{G}{h_0}\left(u_1 - u_2 - \frac{h_1}{2}\beta_1 - \frac{h_2}{2}\beta_2\right)_{x_1}. \tag{A11}$$

Combining eqns (A10, A11) and (A8, A9), results in eqn (40)

$$\left(\frac{dN_{1III}}{dx}\right)_{x_2^+} - \left(\frac{dN_{1I}}{dx}\right)_{x_1^-} = \frac{G}{h_0}\left(\int_{x_1}^{x_2}\left(C_1 N_{1II} - \frac{h_1}{2} D_1 M_{1II}\right)dx - \int_{x_1}^{x_2}\left(C_2 N_{2II} - \frac{h_2}{2} D_2 M_{2II}\right)dx\right). \tag{A12}$$

To derive eqn (41), a similar scheme (as discussed above) can be used. Equilibrium eqn (5), can be combined with eqns (A8, A9) and (A10, A11), which results in the following

$$\left(\frac{dM_1}{dx}\right)_{x_1^-} = \left[Q_1 - \frac{h_1 + h_0}{2}\frac{G}{h_0}\left(u_1 - u_2 - \frac{h_1}{2}\beta_1 - \frac{h_2}{2}\beta_2\right)\right]_{x_1} \tag{A13}$$

and similarly

$$\left(\frac{dM_1}{dx}\right)_{x_2^+} = \left[Q_1 - \frac{h_1 + h_0}{2}\frac{G}{h_0}\left(u_1 - u_2 - \frac{h_1}{2}\beta_1 - \frac{h_2}{2}\beta_2\right)\right]_{x_2}. \tag{A14}$$

Subtracting eqn (A13) from eqn (A14) and using eqns (A8–A11) yields eqn (41).